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A model of stage change explains the average rate of stage of development and its relationship to the predicted average stage ("smarts")

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ABSTRACT

A number of different previous methods for measuring "smarts" have led to the model of hierarchical complexity (MHC), a context free neo-Piagetian mathematical model of behavioral complexity. It provides a way to classify tasks as to their hierarchical complexity. Using the model of hierarchical complexity, this study examines how differences in rate of stage change results in a difference in the highest average stage (smarts") attained by 70 year old adults. The average stage of development ("smarts") was shown to be predicted by the log of age with an $r = .79$. It uses data from Colby, Kohlberg, Gibbs, Lieberman (1983) to test the model. It also predicts that on the average there is one stage of development during adulthood.

KEYWORDS: stage limits, logage, age, IQ, smarts, smartness, adult stages

There has been a long controversy about the relationship between "smarts" and biology and the environment. Most typically, the "smarts" being discussed in here has been measured using IQ. There are extensive reviews of this controversy so it will not be covered in this paper. Also there is an extensive literature on the relationship between stage and IQ (DeVries, 1974; Dudek, Lester, & Goldberg, 1969; Humphreys & Parsons 1979; Kohlberg & Devries, 1984; McClelland, 1973). In general, these studies have shown the relationship between stage and IQ to be modest. The problem with all of these studies is that only a small range of ages and stages have been used. This limitation would tend to attenuate the correlation between stage and IQ.

Among the problems with IQ as a measure of smarts, a major one is that there has been no psychophysical approach. IQ is based on a psychometric approach. Psychometrics depends on an analysis of only responses to items without any apriori theory about the meaning of the items or stimuli that lead to those responses. This kind of approach makes it difficult to interpret many results, because the nature of the items or stimuli, and the relationship between them, are not specified in advance. Psychophysics, on the other hand, depends on finding the relationship between independently scaled characteristics of stimuli and the responses to those stimuli. Consider that sound intensity is the scaled physical property of sound. The reported loudness is the response to the intensity of those sounds. One of the major problems with IQ is that we do not have an a-prior scaled difficulty of the IQ items. One could find out how difficult participants found the items based on their performance. In contrast to psychophysics, psychometrics was introduced when there was no understanding of how to scale the difficulty of items beforehand. As a result of the limitations of psychometrics in the case of IQ, it is not possible to figure out why the items in IQ tests have the difficulties they do. It is also not possible to explain why the items then "loaded" on the general factor. That is, why certain items are solved earlier and other items later can only be explained in a post hoc manner.

The problems with a number of different previous methods for measuring "smarts", but particularly the IQ measures, have led to the model of hierarchical complexity (MHC). The MHC is a psychophysical as opposed to a psychometric approach to the measure of "smarts." The order of hierarchical complexity is a property of the tasks that exists in the real world and is defined *a priori*. The order of hierarchical complexity (OHC) is an analytic measure applied to tasks. An organism's performance on those tasks is called *stage*. It should be noted here that this theory replaces earlier notions of stage (Inhelder & Piaget, 1958). Those notions, in a manner similar to IQ, confounded the stimulus and response

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in assessing stage. The vast majority of stage-related data have been generated by simply scoring responses and ignoring the task or stimulus.

When stage is based on the *a priori* analytic scaling of items that is possible using the model of hierarchical complexity, stage does not have the problems of the psychometric method. Rather, as will be spelled out, the "Order of hierarchical complexity (OHC)" of a task predicts the performance on those tasks. The performance is measured by the Rasch scaled difficulty of the items (Commons, Goodheart, Pekker, Dawson, Draney, & Adams, 2008). Relating the order to the task performance in this way is a use of psychophysics not psychometrics.

There are several other advantages of the use of MHC to study "smarts." One important one is that the order of hierarchical complexity has been shown to be a property of tasks that is independent of the form, content and method of testing. It is not based on an analysis of performance but on an analysis of task demands. By using Rasch analysis of participant's performance, the obtained difficulty has been compared to the posited order of hierarchical complexity. The OHC predicts the Rasch scores with an *r* between .91 and .98, as shown in Giri, Commons and Harrigan (2014). This study also showed that factoring nine stage measures differing in form, content, and method yielded only one factor, stage.The implications of these results that the model of hierarchical complexity can be used with any kind of task from any domain.

A second important advantage is that the MHC includes stages of action and reasoning beyond the formal stage. Using the MHC to *a priori* and analytically determine the difficulty of the items on a typical IQ test (the WAIS-V), suggested that these items top out at the formal order. There were only a very small number of items characteristic of the next order, systematic (Chen, & Commons, 2014). Not only is the IQ test limited to having only a few items that require the first stage beyond the formal stage, it also does not go down below the primary stage. This was also determined from the above-mentioned coding of the items on the WAIS (Chen, & Commons, 2014). What is particularly problematic, since Commons (2008) estimated that about 20% of educated adults complete tasks that are more complex than the single variable problems that are required for the formal stage (Commons, Miller & Kuhn, 1982) is the relative lack of items that test for reasoning that is more complex than the formal order. The next order, systematic, involves successfully solving dilemmas or problems with two or more independent or "causal" variables.

The purpose of this paper is to see how the rate of development predicts stage attained at any age. The use of the model of hierarchical complexity allows for a dynamic understanding of the accumulation of "smarts" over a lifetime. The paper does this by testing the relationship of stage of development to age. The longitudinal data set to be used was collected by Colby, Kohlberg, Gibbs and Lieberman (1983) using the moral maturity scale. The moral maturity scores were converted into hierarchical complexity stage scores (Tuladhar & Commons, 2014),

Table 1. order number and order name

order		moral maturity scores (MMS)
number	name	
0	computational	—
1	automatic	–150
2	sensory or motor	–100
3	circular sensory motor	–50
4	sensory-motor	0
5	nominal	50
6	sentential	100
7	preoperational	150
8	primary	200
9	concrete	250
10	abstract	300
11	formal	350
12	systematic	400
13	metasystematic	450
14	paradigmatic	500
15	crossparadigmatic	550
16	meta-crossparadigmatic	600

Note. adapted from "Correspondence between some life-span, stage" by Commons, M. L. & Tuladhar, C. T., 2014, *Behavioral Development Bulletin*, *19*(3), p. 26.

First, in order to allow for a better understanding of this material, the paper introduces the model of hierarchical complexity. It then presents a stage acquisition model of the factors necessary for development to take place. Third, a statistical model that shows the relationship between stage and maturation is presented. In that model, age represents maturation. Maturation is mostly due to biology but may be affected by environmental factors, including stimulation, nutrition and disease. What is to be examined here is how good age is at predicting stage is. Fourth, based on the relationship between stage and age, does stage progress at a diminishing rate with age?

» THE MODEL OF HIERARCHICAL COMPLEXITY

The model of hierarchical complexity (MHC) is a non-mentalistic, neo Piagetian mathematical model (Krantz, Luce, Suppes, & Tversky, 1971; Luce & Tukey, 1964). MHC deconstructs tasks into the actions that must be done at each order and this allows for the measurement of stage performance. MHC provides an analytic and *a priori* measurement of the difficulty of task actions and postulates that the difficulty is represented by the orders of hierarchical complexity (OHC) (Commons & Pekker, 2008). There are 17 known orders of hierarchical complexity. This is shown in Table 1, along with the corresponding Moral Maturity Scale scores (Colby & Kohlberg, 1987A, 1987B; Colby, Kohlberg, Gibbs, & Lieberman, 1983). Tuladhar and Commons (2014) have described the one to one relationship between moral maturity scores and MHC stages. Moral maturity scores are stages in the moral judgment subdomain.

Hierarchical complexity refers to the number of times that the coordinating actions must organize lower order actions. The hierarchical complexity of an action is determined by decomposing the action into the two or more simpler actions that make it up. This iterative process is done until the organization can only be carried out on a set of simple elements that are not built out of other actions. As shown in Figure 1, actions at a higher order of hierarchical complexity: *1)* are defined in terms of actions at the next lower order of hierarchical complexity; *2)* organize and transform the lower-order actions; *3)* produce organizations of lower-order actions that are new and not arbitrary, and cannot be accomplished by those lower-order actions alone. Once these conditions have been met, the higher-order action coordinates the actions of the next lower order.

To illustrate how lower actions get organized into more hierarchically complex actions, consider a simple example. Completing the entire operation 3 × (4 + 1) constitutes a task requiring the distributive action. That distributive action non-arbitrarily orders adding and multiplying to coordinate them. The distributive action is therefore one order more hierarchically complex than the acts of adding and multiplying alone; it indicates the singular proper sequence of the simpler actions. Although simply adding can result in the same answer, people who can do both display a greater freedom of action. Thus, the order of complexity of the task is determined through analyzing the demands of each task by breaking it down into its constituent parts. For example, an order- three task can be broken down into a sequence of three concatenation operations. A task action of order three operates on two or more task actions of order two. A task action of order two operates on two or more task actions of order one.

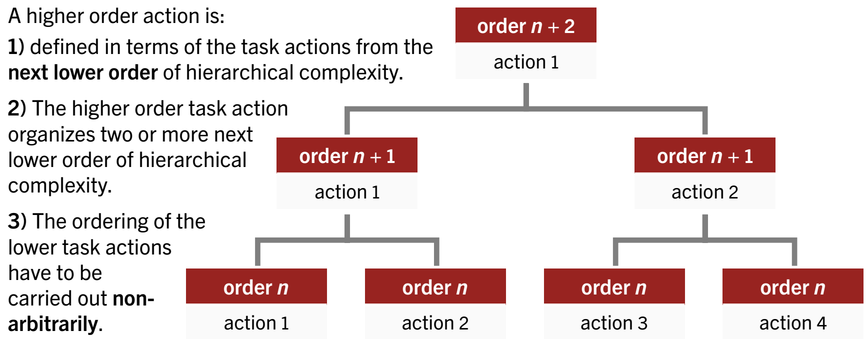

Figure 1. Shows the order of hierarchical complexity tree. Each higher order action organizes two or more next lower order actions. The hierarchical complexity of a task refers to the number of concatenation operations it contains, that is, what is the number of recursions that the coordinating actions must perform?

Each task difficulty has an order of hierarchical complexity required to complete it correctly. Because tasks of a given order of hierarchical complexity require actions with the matching stage number to perform them, the stage of the participant's performance is equivalent to the order of complexity of the successfully completed task. Tasks are also quantal in nature. They are either completed correctly or not completed at all. There is no intermediate state. For this reason, the model characterizes all stages as hard and distinct. The quantal feature of tasks is thus particularly instrumental in stage assessment because the scores obtained for stages are likewise discrete.

In considering questions, such as the relationship between "smarts" and biology, or other questions related to "smarts", the model of hierarchical complexity presents several advantages. First, the three axioms (See Figure 1) make it possible for the model's application to meet real world requirements, including the empirical and analytic. In particular, earlier uses of arbitrary organizations of lower order of complexity actions, possible in the Piagetian theory, despite the hierarchical definition structure, leaves the functional correlates of the interrelationships of tasks of differential complexity ill-defined. Moreover, this model is consistent with the neo-Piagetian theories of cognitive development. According to these theories (e.g. Pascual Leone, 1970), progression to higher stages or levels of cognitive development is caused by increases in processing efficiency and working memory capacity. That is, higher-order stages place increasingly higher demands on these functions of information processing, so that their order of appearance reflects the information processing possibilities at successive ages. Finally, the MHC can be applied to any content and the behavior of any organism, not just humans.

» THE RELATIVE ROLES OF MATURATION AND ENVIRONMENTAL CONTRIBUTIONS TO SMARTNESS ("SMARTS")

There are two main forms of stage change, which are developmental advances in behavior. Horizontal décalage refers to fact that once a certain task of a given order may be completed correctly in one task sequence, it is not necessarily initially completed correctly in another task sequence. In Baylor's (1975) words, "a horizontal décalage arises when a cognitive structure that can be successfully applied to task X cannot, though it is composed of the same organization of logical operations, be extended to task Y." One kind of stage change, therefore, is simply learning to apply a set of actions learned in one domain to another set of actions, at a similar order of hierarchical complexity, in another domain. Most developmental advances in behavior are of the horizontal form. In contrast, there is also real stage change, also sometimes called vertical décalage. This paper focuses on the vertical form.

From our perspective, there are two main factors necessary to developmentally advance behavior. They are conceptually separable but both are necessary. First is that there has to be a capacity to change. Second, the environment must be supportive of change.

1. There must be a "capacity" to change. This may be represented most compactly by Pascual-Leone's (1970) suggestion that to solve a problem at order N, there has to be a working memory of 2^N. This is what is termed capacity. This means that there are limits to what the environment produces in development at any age. Also the environment has to be tuned to the present performance of the organism to produce "maximal" stage change. No matter how much and how good the training is, at a given age there is an upper limit of stage that may be attained. Note that what can be trained at a given age varies across individuals. Capacity is assessed by finding where in a developmental sequence the organism is performing. There is ideal capacity. Ideal capacity is what would develop if the environment were perfectly tuned to maximize development. Capacity is also dynamic, changing with age and experience.

2. There must be contingent reinforcement for engaging in the task. There are two parts to this.

a) First, the task has to be appropriate for where the organism is performing in the sequence of tasks. For a task to be appropriate, it has to have some reasonable probability of being completed successfully. This requires first that the task is correctly placed in the developmental sequence. Second, the organism has to be functioning at that place in the sequence. One can overwhelm a student by giving tasks that are too much above the present stage at which they are functioning. One can bore a student by giving them work they already do perfectly.

b) Second, there must be some kind of reinforcing consequence that ensues from the completion of the task. The reinforcement is conceived of in much broader terms than what behavior analysts generally use. Reinforcement can include task mastery, which is set up by the drive of being interested, which is usually described as curiosity. Reinforcement may also be social recognition and attention.

The argument to be presented here does not suggest that there is not a contribution of environment to "smarts". With reinforcement for correct answers in the laundry problem, 5th and 6th graders moved from 100% reasoning at the concrete stage to 75% reasoning at the formal stage (Commons, Davidson & Grotzer, 2007). Our preliminary data on training with reinforcements in a non-literate Nepal sample shows a great increase in formal stage answering and even some systematic stage answering (Commons, Giri, & Tuladhar, 2013). These studies show that environment did contribute to smarts. But for the environment to contribute to smarts there has to be reinforced training. There also has to be transfer of training tests to exclude the possibility of rote learning.

» ENVIRONMENTALLY EFFICIENT METHOD OF MAKING CHANGES IN BEHAVIOR STAGE

Based on the above discussion, we propose the following formal model of the factors necessary for behavioral stage change. The change in behavioral stage, ΔB, is simply the product of the time actively engaged in getting the right answers to a task when placed in the developmental sequence correctly. This is shown in the following equation:

$$\Delta_B = t \times pl, \quad (1)$$

where ΔB is the change in behavior, $t = f_1(S_{\text{Contingencies of reinforcement for correct answers}})$, and $pl = f_2$(being placed in the right place in the developmental sequence).

The more time spent alive inevitably leads to more information being processed. More information being processed leads to higher chances of coordinating the information to higher stages. Time engaging actively on a task is sensitive to contingent reinforcement of correct responses.

Consider the probability of a 7-year old effectively engaged in the task of solving the equation $x - 1 = 0$. Most children at 7 years of age generally perform at the primary stage 8. This child would also most likely be performing at the primary stage, two stages before the formal stage 11. Because this is a formal order 11 task, the child will likely fail it. So a task being placed too high in a task sequence leads to failure if the participant is not performing at the same stage and substage in the developmental sequence. Similarly, a task being placed too low on the task sequence leads to boredom on the part of the participant, and a failure to engage in the task. Because the person already does the task, learning does not occur as well.

» MEASURING THE AVERAGE STAGE OF DEVELOPMENT (SMARTS) USING MHC AS A FUNCTION OF AGE

As argued above, behavioral stage as measured by the model of hierarchical complexity gives a relatively unbiased measure of what we call "smarts." After establishing the usefulness of stage as a measure of "smarts", this paper presents a general and simple notion, that what produces behavior change is the amount of time spent engaged with environmental tasks presented at an appropriate order of complexity and accompanied by effective reinforcing events. This is the ΔB equation discussed above. This is a very general notion. Because it is based on the MHC, it also is not based on specific context or content or any specific intervention. It can apply to any organism at any order of development. Even though there is evidence that measured behavioral stage may be increased over time based on experience and maturation, there still may be limits as to how far training can get. Therefore a second and separate question that can be pursued here is what are the constraints to development at a given age?

The next part of this paper uses some empirical data to see what the best predictor of "smarts" or the organism's currently measured behavioral stage is. We start with the simplest model, which examines the relationship between stage ("smarts") and age (maturation). The explanation for the relationship between stage ("smarts") and age (the maturational contribution) is simple. Individuals develop as long as they are appropriately stimulated and supported by the environment, as shown in the ΔB equation discussed above. They also have to have enough capacity to engage with increasingly hierarchically complex tasks. As long as more complex problems and dilemmas are presented by the environment, there will be an increase in stage under these conditions. In other words, consider age, stage, and the order of hierarchical of complexity of the tasks presented, the model of hierarchical complexity provides an explanation for how stage change results in average stage attained at a given age.

Four steps show what determines the stage attained

The steps of the derivation of the regression equation for predicting the average stage attained by the participants as a function of age is shown next.

Step 1: recall that there are the differences between the order of hierarchical complexity of tasks and the corresponding stage of performance on those tasks. Order of hierarchical complexity (OHC) is an *a priori* analytic measure of difficulty applied to tasks. Stage is a performance measure of the most hierarchically complex task solved by the organism in question.

Step 2: Based on the information in Step 1, it is important to start out by deriving a measure of stage that can be predicted. This starts with the following definition.

$$\text{Total amount of hierarchical complexity of a task} = 2^N_{OHC}.$$

This is based on Pascual-Leone's (1970) suggestion that to solve a problem at order N, there needs to be a working memory of 2^N. Due to the definitions of stage and of order given in Step 1, the stage number, N_{stage}, is the same as the number N_{OHC}, for the most hierarchically complex task solved

$$2^N_{OHC} = 2^N_{stage}.$$

Therefore, what we will be predicting is N_{stage} or performance. Individual scores will be predicted. The predictions for individual scores will include all valid data for all participants.

Step 3: Determine how age will be considered. The prediction is that *Age* = *t* helps determine the amount of hierarchically complex information processed correctly. As asserted in Step 1, stage is the order of hierarchical complexity of a task (OHC) completed correctly. But using age in terms of simple number of years would not be appropriate as explained next. It is asserted and to be tested that $N_{stage} = \log_2(t)$. This equation shows that the more time spent alive inevitably leads to more information being processed correctly. What is to be predicted is N_{stage}, which is

$$\log_2\left(2^N_{stage}\right),\ N_{stage} = f_0\left(\log_2\left(2^N_{stage}\right)\right).$$

Because we take the $\log_2(2^N)$ to get stage, we have to take the $\log_2$ of age: $\log_2$(age). We only explore one predictor, $\log_2$(age) to see whether it is a good predictor of behavioral stage.

Step 4: The more information being processed at a given time leads to higher chances of coordinating the information at higher orders of hierarchical complexity. Therefore, at the fourth step, it will be shown that development is set by the rate of stage change, the parameter K. The parameter K, by definition, is the rate of change of stage with age. The rate of change is represented as a partial derivative noted as ∂. The derivative of stage $\partial N/\partial t$ with respect to time is

$$\frac{\partial}{\partial t}\left(\log_2\left(2^N\right)\right) = K.$$

Then N_{stage} is substituted for $\log_2(2^N)$ yielding

$$\frac{\partial}{\partial t}\left(N_{stage}\right) = K.$$

By using N_{stage} as a stage variable, K may be found for individuals or for groups of individuals. In sum, this is:

$$K = \frac{\partial}{\partial t}\left(N_{stage}\right),$$

which is the partial derivative of N_{stage} with respect to t. Because N_{stage} is a function of age, K, which may be found for an individual or for groups, requires the age of the individual or the average age of the group.

Table 2. mean and *SD* for the variables

mean (*SD*) MMS	mean (*SD*) stage (MHC)	mean (*SD*) age	mean $\log_2$age (*SD*)
M = 306.78; *SD* = 71.47	*M* = 10.13; *SD* = 1.15	*M* = 20.9; *SD* = 7.51	*M* = 1.27; *SD* = .19

» MORAL MATURITY SCORE WAS REGRESSED ON LOG₂(AGE) OF PARTICIPANTS

In this paper we look at individual Moral Maturity Scores and the corresponding stage from the model of hierarchical complexity. We will argue that when one looks at individuals, there are two contributions to Moral Maturity Scores: ***a)*** the maturational and ***b)*** the environmental. Training, as shown by ΔB is one of the contributors to individual differences and behavioral change up until the limits imposed by maturation.

The data used in this analysis come from Colby, Kohlberg, Gibbs, Lieberman (1983). They presented three longitudinal studies on Moral Judgment. They reported individual Moral Maturity Scores. The data set consists of 51 people with multiple observations per person of their moral maturity assessed at approximately four year intervals. There were 225 observations in total. Some participants are missing some assessments.

» STATISTICAL ANALYSIS

The purpose of this section is to explain the statistical analysis employed to estimate how moral maturity is related to the logarithm, base 2 of the age of person at time of assessment.

First, to make clear that moral maturity scores (MMS) have an equivalent stage of development as the model of hierarchical complexity stage score, we show the simple linear relationship between the two (Tuladhar, & Commons, 2014). The following simple equation represents that result.

$$\text{Behavioral developmental stage} = 4 + 0.02\text{MMS}$$

Second, the moral maturity at an age of assessment is specified as linearly related to the logarithm, base 2, of age of the individual at assessment, as in Equation 2. The equation to be fitted is Equation 2

$$\text{MMS}(t) = \beta_0 + \beta_1 \log_2(\text{age}) + \varepsilon(t), \quad (2)$$

where t is the age of an observation at time of assessment, $\log_2$(age) denotes the base 2 logarithm of the age of an individual at assessment, β_o denotes the scale constant relating the offset between Moral Maturity Scores and $\log_2$(age), and β_1 denotes the slope. It is the slope that we get from the regression between MMS and $\log_2$(age). Additionally, $\varepsilon(t)$ is an error term, whose expected value is zero and whose variance is σ^2_ε, and ρ (Rho Greek letter) denotes the correlation between two temporally adjacent test scores for the same person.

The dependent variable MMS(t) denotes an interval level measure of moral maturity at the assessment that occurred at age t.

See the Appendix A for definitions of terms, and a more detailed explanation of the statistical model.

» RESULTS

Figure 2 shows the scatter plot of the data. Full information maximum likelihood estimates of the parameters of Equation (2) account for the effects of the repeated measurements and missing measures (See Appendix A). Table 3 (which follows below) presents the maximum likelihood parameter estimates and their statistical significance. Average moral maturity scores were predicted by $\log_2$(age): $r(225) = 0.785$; the adjusted $R^2 = 0.6163$, which is the proportion of the variance explained by the model; $p = .00014$ and the non-adjusted $R^2 = 0.6214$.

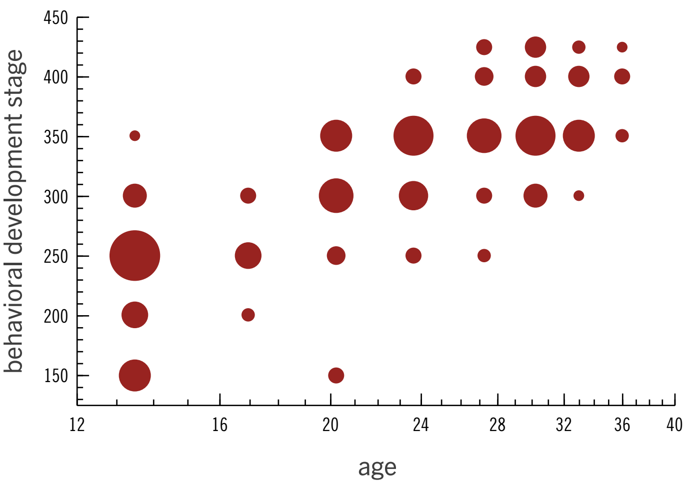

Figure 2. Shows the scatter plot for individuals. The age of participants was plotted on the *x*-axis ($\log_2$ scaled) and behavioral developmental stage was plotted on the *y*-axis. The size of the circles represents the number of cases.

From above and Table 3,

$$\text{MMS} = -73.92 + 88.17\log_2(\text{age}) + \varepsilon(t), \tag{7}$$

where the expected value of $\varepsilon(t) = 0$.

From above, we have the equation showing the simple linear relationship between moral maturity scores (MMS) and stages of development,

$$\text{Behavioral development stage}(t) = 4 + 0.02\text{MMS}(t). \tag{8}$$

Substituting the MMS into that equation yields

$$\begin{aligned} &= 4 + 0.02\left(-73.92 + 88.17\log_2(\text{age})\right) \\ &= 4 - 1.48 + 1.76\log_2(\text{age}) \\ &= 2.52 + 1.76\log_2(\text{age}). \end{aligned} \tag{8}$$

Next, the values of K are derived. As said previously K is the rate of change of stage with age. From Appendix C, the estimated value of K was found for $\delta(N_{\text{stage}})/\delta$, the partial derivative of N_{stage} with respect to t.

$$\text{Hence,}\quad K = \frac{\partial}{\partial t}(N_{stage}) = \frac{2.539}{\partial(\text{age})}. \quad \text{(see Appendix C)}$$

» RELATIONSHIP BETWEEN STAGE AND AGE

In this section, it will be shown how small changes in β_1 result in large changes in stage, given age. This is because the long term cumulated effect changes the ultimate stage a person may reach. Note, that at any time, the predicted average stage is increasing. This is roughly a somewhat similar argument to IQ and the rate of learning. But it differs in that there is no arbitrary end, due to the measurement instrument, as seen with IQ. That is because MHC tests start in infancy and go through to the end of adulthood. The laundry task, for example, starts in the circular sensory-motor stage 3 (age .8) and goes through to metasystematic stage 13.

Using Equation 9, the predicted stages across ages are shown in Figure 3, given different changes in values assumed for the value of β_1.The first change is to make a 10% increase in β_1 and then a 20% increase in β_1. Generally, each stage is one standard deviation from the next one. In IQ, 15 points is one standard deviation. Depending on age, 10% will predict a different standard deviation for stage as seen in Figure 3. The second change is to make a 10% decrease in β_1 and then a 20% decrease in β_1. See appendix B for the mathematics underlying these results. What Figure 3 shows is that as people get older, the maximum stage is never attained. That is, the slope of the lines flattens out as one ages.

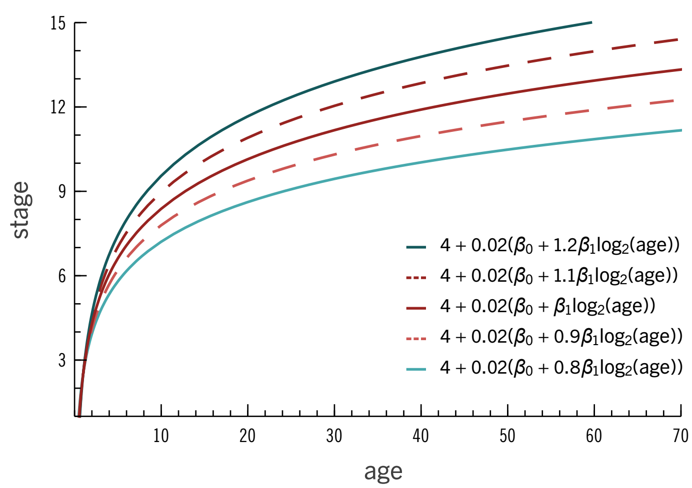

Figure 3. the relationship between age and stage

Table 3. model parameter estimates and statistical significance

parameter	parameter estimate	standard error of estimate	*t*-value	*p*-value
β_0	-73.92	19.03	-3.88	0.00014
β_1	88.17	4.41	20.00	1.79×10^{-51}
σ_v^*	39.09	1.91	20.52	0.0
ρ	0.46	0.07	6.91	4.99×10^{-11}

*Note. * represents 1 tailed test*

This 'slowing down' in development is shown in terms of specific predicted stages for each decade in adulthood in Table 4. While the average change in predicted stage across these 5 decades is .63, it can be seen that stage change in the decade from age 20 to age 30 is predicted to be twice what it is in the decade between 60 and 70.

Table 4. predicted stages for different adult ages (for the average case)

age	predicted stage	change in predicted stage
20	10.13	—
30	11.16	1.03
40	11.89	0.73
50	12.45	0.56
60	12.92	0.47
70	13.31	0.39

Table 2 showed that at the average age of the participants, which is 20.9 years, the average stage is around the Abstract Stage, 10.1. Table 5 shows that at the age of 8, the average stage is 7.6 (transitional to Primary 8). On the other hand, if one is in the top 20 percent, the stage of 8.8 is already transitional to concrete, a standard deviation above the mean. Note also that this small difference at age 8 becomes a large difference at age 70. If one is in the top 20 percent, one reaches the Crossparadigmatic stage 15, which is an overestimation from the model. This is because there is over extrapolation from the data. If one is down 20 percent, one only reaches the abstract stage 10, which makes sense.

Table 5. stage equivalent at age 8 and age 70 with 10 and 20 percent increase in β_1 and 10 and 20 percent decrease in β_1

age	average stage equivalence	stage with 10% increase in β_1	stage with 20% increase in β_1	stage with 10% decrease in β_1	stage with 20% decrease in β_1
8	7.6	8.2	8.8	6.2	6.8
70	13.1	14.2	15.1	11.9	10.9

» RELATIONSHIP BETWEEN CHANGE IN STAGE AGE DUE TO CHANGE IN AGE

This section will show the relationship between changes in stage due to changes in age (see Figure 4) under the same conditions as above: First the relationship is shown with a 10 percent increase in the estimated β_1 and then a 20 percent increase in the estimated β_1. Second, the relationship between age and the change in stage is shown due an increase in age with a 10 percent decrease in the estimated β_1 and then a 20 percent decrease in the estimated β_1. See Appendix C for the math and the model.

The dynamic representation of change in stage changes at different points in the lifespan. Essentially the rate of change decreases with age. What is somewhat startling is how much a difference the estimated slope coefficient makes on that rate of change.

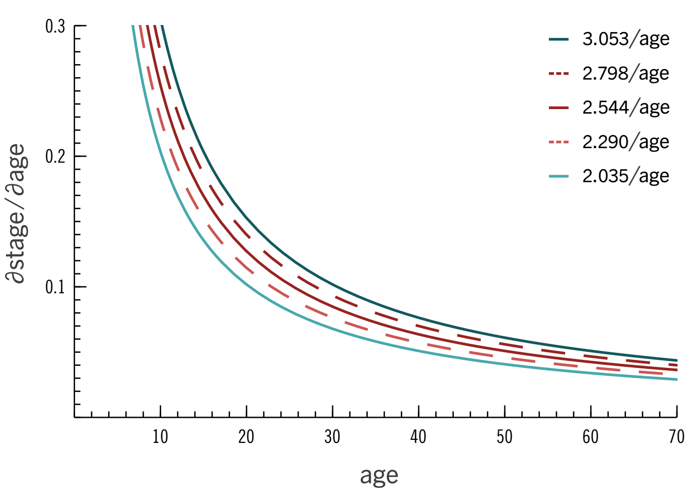

Figure 4. This plot shows the relationship between age and change in stage. These expressions are the derivatives of the expressions in Figure 3 with respect to age.

» DISCUSSION

We have argued that the idea that "smarts" can be measured with behavioral stage. Stage is a better measure than something like IQ because with the model of hierarchical complexity, the *a priori* difficulty of the items is known. This makes it possible to understand what the results mean in terms of what exactly was being tested.

Second, using notions from MHC, one can then construct powerful models of what predicts development. Specifically, this study presented an empirical test of a mathematical model of the average attained stage of development ("smarts"). It uses data from Colby, Kohlberg, Gibbs, Lieberman (1983) to test the model.

The first major result of this approach is the demonstration that stage increases as function of $\log_2$(age). The correlation between age and stage was .79, accounting for a surprisingly large amount of the variance in stage (61%).

A second major result is that as people get older, one can see that a maximum stage is never attained. Here we proposed that based on a single parameter, κ, MHC provides an explanation for how differences in rate of stage change result in a difference in average stage, given age. This parameter, κ, is equal to the derivative of moral maturity stage of a person's performance with respect to time. It is suggested that κ is a measurement of increase in "smarts" as age increases. As age increases, the value of κ decreases and vice versa. What is constant throughout the life span as far as our limited data shows is the product of κ and age, 2.539. This finding is because the increase in stage is less and less with increasing age.

A third major result was the finding that small changes in β_1 (slope from the regression between stage and $\log_2$(age)) results in an increasingly large change in stage. This is because the long-term cumulated effect changes the ultimate stage a person may reach.

A fourth major result is that, on the average, there is less than one stage of development predicted during each decade of adulthood.

Each of these findings has important implications for the study of development, and particularly in the areas of aging and positive adult development.

Stage of development is the best measure of "smarts" and should replace IQ for individuals (Commons & Ross, 2008). It only loads on a single factor no matter the content or context of the instrument used to measure it, as shown by Giri, Commons, & Harrigan (2014). Because age is such a strong predictor of stage, people should consider the maturational contribution to stage of development.

Stage measurement has problems however. It is sensitive to a number of things. These include the reinforcement history for giving the highest stage performances on tasks. Many traditional cultures either punish high stage responses or at least do not reinforce them (Commons, Galaz-Fontes &Morse, 2006; Commons, Giri & Tuladhar, 2014; Day, 2008). This is the case in all subdomains even though we find only find one overall domain; stage (Giri, Commons, & Harrigan, 2014). For example in Ravnican's (2013) study, people gave traditional and concrete stage 9 answers to the version of the laundry problem she used. Day (2008) found that religious fundamentalists gave concrete stage 9 answers to the saying "he who is without sin, cast the first stone" and other moral dilemmas. This extended to other moral stage answers. By using training, one can overcome these limitations, as we are showing in a study in Nepal (Commons et al., 2013). The main solution is to use simpler stage tasks and reinforce correct answers and then test with a similar set of tasks.

The importance of this paper is twofold. First, this paper addresses the issue of talking about individual differences of smartness with one parameter. This parameter is *K*, the rate of change of stage with Age. The behavioral sciences have tried using IQ tried to explain individual differences but IQ does not explain this well because there was no *a priori* idea of what the difficulty of the IQ items were. This made it extremely difficult to interpret what exactly IQ measured and therefore what the individual differences could be attributed to. Second, stage has a clear notion that attempted task solutions are different at each stage, with the higher stage action building upon the lower stage actions. There is an underlying logic to the progression of answers in the stage sequence. Stage is the equivalent to mental age. Stage divided by age is the equivalent to IQ, with the difference that it can be applied to tasks across the lifespan. By combining stage with age, one derives the average potential to learn. ■

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APPENDIX

» APPENDIX A

Parameters estimates

It is postulated that the errors arise from a first-order autoregressive process known as the AR-1 model. An autoregressive (AR) model is a representation of a type of random process. As such, it describes certain time-varying processes in nature, economics, etc. The autoregressive model specifies that the error depends linearly on its own previous values. In the first autoregressive model, AR-1 model, the error in equation [2] at time t is correlated with the error at time t - 1, as in equation [3],

$\varepsilon(t) = \rho\varepsilon[t - 1] + v(t)$ [3]

Where,

$\varepsilon(t)$ = the error at time t

ρ (Rho Greek letter) denotes the correlation between two temporally adjacent test scores for the same person.

$v(t)$ is an error term whose expected value is zero and whose variance is σ_v^2

If ε is a column vector of errors, $\{\varepsilon[1], \varepsilon[2], ..., \varepsilon[n]\}^T$, then the covariance of the errors in the model, $\Phi = E[\varepsilon\, \varepsilon']$, for a person with three observations, for example, has the form given by equation [4]. Note, ε is a column vector of errors but is written as a row vector, with T indicating that this row vector is transposed.

$$\varphi = E\left[\varepsilon\, \varepsilon'\right] = \frac{\varphi_v^2}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \qquad (4)$$

Where,

ρ (Rho Greek letter) denotes the correlation between two temporally adjacent test scores for the same person

Φ_v^2 denotes the variance of $v(t)$. It denotes the common variance held in all the observations and all the errors.

Capital σ_v has a matrix form.

Φ denotes the covariance of the errors

$\Phi = E[\varepsilon\, \varepsilon']$ is the model of errors for a person with three observations

Assuming the v(t) errors have normal distributions, the collection of ε errors for an observation will have a Multinormal distribution. Hence, to continue the example of three observations for a person, the probability of the mm(t) and $\log_2$(age) observations are given by equation [5], and how the various parts of it are calculated.

$$\text{PDF}\left[\text{MultinormalDistribution}\left[\{0,0,0\}, \frac{\sigma_v^2}{1-\rho^2}\begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}\right]\right] \quad (5)$$

$\text{mm}[1] - (\beta_0 + \beta_1 \log_2(t[1]))$,

$\text{mm}[2] - (\beta_0 + \beta_1 \log_2(t[2]))$,

$\text{mm}[3] - (\beta_0 + \beta_1 \log_2(t[3]))$

Where,

t = the age of an observation at time of assessment

$t[1]$ = the age of an observation at time 1 of assessment

$t[2]$ = the age of an observation at time 2 of assessment

$t[3]$ = the age of an observation at time 3 of assessment

$\log_2$(age) denotes the logarithm, base 2, of the age of an individual at assessment.

β_0 denotes the scale constant relating the offset between Moral Maturity Scores and $\log_2$(age)

β_1 denotes the slope. It is the slope that we get from the regression between MMS and $\log_2$(age).

σ_v^2 denotes the variance of $v(t)$. See equation 3. It denotes the common variance held in all the observations and all the errors.

$$\frac{1}{2\sqrt{2}\pi^{\frac{3}{2}}\sqrt{\frac{(-1+2\rho^2-\rho^4)\sigma_\gamma^6}{(-1+\rho^2)^3}}}$$

$$= \frac{1}{2}\left(-\frac{\left(-\beta_0 - \frac{\beta_1 \text{Log}[t[1]]}{\text{Log}_2} + mm[1]\right)\left(-\beta_0 \text{Log}_2 + \beta_0\rho\text{Log}_2 - \beta\,\text{Log}[t[1]] + \beta_1\rho\text{Log}[t[2]] + \text{Log}_2\, mm[1] - \rho\text{Log}_2\, mm[2]\right)}{\text{Log}_2\,\sigma_\gamma^2}\right)$$

$$-\frac{\left(-\beta_0 - \frac{\beta_1 \text{Log}[t[2]]}{\text{Log}_2} + mm[2]\right) \times \left(-\beta_0 \text{Log}_2 + 2\beta_0\rho\text{Log}_2 - \beta_0\rho^2\text{Log}_2 + \beta_1\rho\text{Log}[t[1]] - \beta_1\text{Log}[t[2]] - \beta_1\rho^2\text{Log}[t[2]]\right]}{\text{Log}_2\,\sigma_\gamma^2}$$

$$\frac{+\beta_1 \rho \,\mathrm{Log}[t[3]] - \rho \mathrm{Log}_2\, mm[1] + \mathrm{Log}_2\, mm[2] + \rho^2 \mathrm{Log}_2\, mm[2] - \rho\, \mathrm{Log}_2\, mm[3])}{\mathrm{Log}_2\, \sigma_\gamma^2}$$

$$-\frac{\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[3]]}{\mathrm{Log}_2} + mm[3]\right)\left(\left(-\frac{\rho^2\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[1]]}{\mathrm{Log}_2} + mm[1]\right)\sigma_\gamma^2}{1-\rho^2} + \frac{\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[3]]}{\mathrm{Log}_2} + mm[1]\right)\sigma_\gamma^2}{1-\rho^2}\right)\right.}{\frac{\sigma_\gamma^8}{(1-\rho^2)^4} - \frac{2\rho^2\sigma_\gamma^8}{(1-\rho^2)^4} + \frac{\rho^4\sigma_\gamma^8}{(1-\rho^2)^4}}$$

$$\frac{\left(\frac{\sigma_\gamma^4}{(1-\rho^2)^2} - \frac{\rho^2\sigma_\gamma^4}{(1-\rho^2)^2}\right)}{\frac{\sigma_\gamma^8}{(1-\rho^2)^4} - \frac{2\rho^2\sigma_\gamma^8}{(1-\rho^2)^4} + \frac{\rho^4\sigma_\gamma^8}{(1-\rho^2)^4}}$$

$$-\frac{\left(-\frac{\rho\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[1]]}{\mathrm{Log}_2} + mm[1]\right)\sigma_\gamma^2}{1-\rho^2} + \frac{\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[2]]}{\mathrm{Log}_2} + mm[2]\right)\sigma_\gamma^2}{1-\rho^2}\right)\left(\frac{\rho\sigma_\gamma^4}{(1-\rho^2)^2} - \frac{\rho^3\sigma_\gamma^4}{(1-\rho^2)^2}\right)}{\frac{\sigma_\gamma^8}{(1-\rho^2)^4} - \frac{2\rho^2\sigma_\gamma^8}{(1-\rho^2)^4} + \frac{\rho^4\sigma_\gamma^8}{(1-\rho^2)^4}}$$

Consider, now, the problem of representing the probability of the observed values of mm(t) and $\log_2$(age) for a person with missing observations. For example, assume assessments were administered at ages 9, 13, and 17 and the individual is missing the age 13 assessment. The probability density function of the observed errors, $\{\varepsilon[1], \varepsilon[3]\} = \{mm[1] - (\beta_0 + \beta_1 \log_2(t)[1]], mm[3] - (\beta_0 + \beta_1 \log_2(t)[3]]\}$, is found by dropping the expected value of the second error element in the PDF statement and the second row and second column in the covariance matrix. In this case, the probability of the observed assessments is given by equation [5],

Prob[$\{\varepsilon[1], \varepsilon[3]\}$] = ,

$$\mathrm{PDF}\left[\text{Multinornal Distribution}\{0,0\}, \begin{bmatrix} \frac{\sigma_v^2}{1-\rho^2} & \frac{\rho^2\sigma_v^2}{1-\rho^2} \\ \frac{\rho^2\sigma_v^2}{1-\rho^2} & \frac{\sigma_v^2}{1-\rho^2} \end{bmatrix}\right]$$

$$[\{mm[1] - (\beta_0 + \beta_1 \mathrm{Log}_2\, t[1]), mm[3] - (\beta_0 + \beta_1 \mathrm{Log}_2\, t[3])\}]$$

$$\frac{1}{2}\left(-\frac{\left(-\alpha - \frac{\beta \mathrm{Log}[t[3]]}{\mathrm{Log}_2} + mm[3]\right)(-\beta_0 \mathrm{Log}_2 + \beta_0\rho^2 \mathrm{Log}_2 + \beta_1 \rho^2 \mathrm{Log}[t[1]]}{(1+\rho^2)\mathrm{Log}_2\, \sigma_\gamma^2}\right.$$

$$\frac{-\beta_1 \mathrm{Log}[t[3]] - \rho^2 \mathrm{Log}_2\, mm[1] + \mathrm{Log}_2\, mm[3])}{(1+\rho^2)\mathrm{Log}_2\, \sigma_\gamma^2}$$

$$-\frac{\left(-\beta_0 - \frac{\beta_1 \mathrm{Log}[t[1]]}{\mathrm{Log}_2} + mm[1]\right)(-\beta_0 \mathrm{Log}_2 + \alpha\rho^2 \mathrm{Log}_2 - \beta_1 \mathrm{Log}[t[1]]}{(1+\rho^2)\mathrm{Log}_2\, \sigma_\gamma^2}$$

$$\left.\frac{+\beta_1 \rho^2 \mathrm{Log}[t[3]] + \mathrm{Log}[t[3]] + \mathrm{Log}_2\, mm[1] - \rho^2 \mathrm{Log}_2\, mm[3])}{(1+\rho^2)\mathrm{Log}_2\, \sigma_\gamma^2}\right)$$

$$= \frac{1}{2\pi\sqrt{\frac{\sigma_\gamma^4}{(-1+\rho^2)^2} - \frac{\rho^4\sigma_\gamma^4}{(-1+\rho^2)^2}}}$$

Based on these methods, the probability for each person's observed assessments was analytically determined. These probabilities are functions of four parameters, $\{\beta_0, \beta_1, \sigma_v, \rho\}$. The parameters of the problem were estimated with full-information maximum likelihood methods. The log-likelihood of the problem was constructed by taking the logarithm of each probability expression and then these logarithms were added. Gauss-Siedel estimation was used to determine starting values for the parameter estimates (Thisted, 1988). Estimation used *Mathematica*'s Find Maximum command. The estimate of the variance-covariance matrix of the parameter estimates is based on mathStatica's Hessian command (Rose & Smith, 2011; mathStatica, 2011).

» APPENDIX B

This section shows the effect of how small changes in β_1.The first change are a 10% increase in β_1 and then a 20% increase in β_1.

β_1 plus 10% = $(1 + 0.1)\beta_1$

β_1 plus 20% = $(1 + 0.2)\beta_1$

Where,

$\beta_1 = 88.17$

β_1 plus 10% = 96.99

β_1 plus 20% = 105.80

So, in terms of relationship between age and stage with a 10 percent increase in the estimated $\beta 1$

Stage = 4.00 + 0.02 MMS

Stage = 4 + 0.02(-73.92 + β_1 plus 10% *$\log_2$(age))

In terms of relationship between age and stage with a 20 percent increase in the estimated $\beta 1$

Stage = 4 + 0.02(-73.92 + β_1 plus 20% *$\log_2$(age))

The second change is to make a 10% decrease in β_1 and then a 20% decrease in β_1.

β_1 minus 10% = $(1 + 0.1)\beta_1$

β_1 minus 20% = $(1 + 0.2)\beta_1$

Where,

$\beta_1 = 88.17$

β_1 minus 10% = 79.35

β_1 minus 20% = 70.54

So, in terms of relationship between age and stage with a 10 percent decrease in the estimated $\beta 1$

Stage = 4.00 + 0.02 MMS

Stage = 4 + 0.02(-73.92 + β_1 minus 10% *$\log_2$(age))

In terms of relationship between age and stage with a 20 percent increase in the estimated $\beta 1$

Stage = 4 + 0.02(-73.92 + β_1 minus 20% *$\log_2$(age))

» APPENDIX C

This section shows the relationship between change in stage and age. First the relationship between the change in stage and age is shown due an increase in age with a 10 percent increase in the estimated β_1 and then a 20 percent increase in the estimated β_1.

Stage = 4.00 + 0.02 $(\beta_0 + \beta_1 \log_2(\text{age}))$

κ = D[Stage, age]

Where, D is the partial derivative of stage with respective to age, i.e. partial derivative of

$(\beta_0 + \beta_1 \log_2(\text{age}))$

$$= \frac{0.0288539\beta_1}{\text{age}}$$

$$= 0.02\left(\beta_0 + \beta_1 \log_2(\text{age})\right)$$

$$= \frac{2.539}{\text{age}}$$

So in terms of relationship between age and the change in stage due an increase in age with a 10 percent increase in the estimated β_1

Stage = 4.00 + 0.02 $(\beta_0 + \beta_1 \log_2(\text{age}))$

$$\text{Change in Stage} = 4.00 + 0.02\frac{0.0288539\,(\beta_1 + 10\%)}{\text{age}}$$

Second, the relationship between age and the change in stage is shown due an increase in age with a 10 percent decrease in the estimated β_1 and then a 20 percent decrease in the estimated β_1.

In terms of relationship between age and the change in stage due an increase in age with a 10 percent increase in the estimated β_1

$$\text{Change in Stage} = 4.00 + 0.02\frac{0.0288539\,(\beta_1 + 20\%)}{\text{age}}$$

In terms of relationship between age and the change in stage due an increase in age with a 10 percent decrease in the estimated β_1

$$\text{Change in Stage} = 4.00 + 0.02\frac{0.0288539\,(\beta_1 - 10\%)}{\text{age}}$$

In terms of relationship between age and the change in stage due an increase in age with a 20 percent decrease in the estimated β_1

$$\text{Change in Stage} = 4.00 + 0.02\frac{0.0288539\,(\beta_1 - 20\%)}{\text{age}}$$